3.9 MECHANICAL ADVANTAGE

One of the major criteria of which a designer must be aware is the ability for a particular mechanism to transmit torque or force. Some mechanisms, such as a gear train, transmit a constant torque ratio between the input and output because there is a constant speed ratio between input and output (see Chap. 7). In a linkage, however, this is not the case. How might we determine a relationship between *force out* and *force in?* Two observations can be made without further analysis.

1. As hinted in the preceding mention of the gear train, the torque ratio is a function of the speed or angular velocity ratio between output and input links of the mechanism.
2. The torque ratio is a function of geometric parameters, which, in the case of a linkage, will generally change during the course of the mechanism motion.

If we assume that a mechanism is a conservative system (i.e., energy losses due to friction, heat, etc., are negligible compared to the total energy transmitted by the system), and if we assume that there are no effects of inertia forces, *power in* (P_{in}) is equal to *power out* (P_{out}) (see Fig. 3.65). Thus the *torque in* times the *angular velocity in* is equal to the *torque out* times the *angular velocity out:*

$$\boxed{\begin{aligned} &P_{in} = T_{in}\omega_{in} = T_{out}\omega_{out} = P_{out} \qquad (3.40)\\ &\text{or,}\\ &P_{in} = \mathbf{F}_{in} \bullet \mathbf{V}_{in} = \mathbf{F}_{out} \bullet \mathbf{V}_{out} = P_{out} \qquad (3.41)\end{aligned}}$$

where T_{in} and F_{in} are torque and force exerted *on* the linkage, and T_{out} and F_{out} are those exerted *by* the linkage; where $\mathbf{V}_{in}$ and $\mathbf{V}_{out}$ are the velocities of the points through which $\mathbf{F}_{in}$ and $\mathbf{F}_{out}$, respectively, act; and where, in vector form,

$$\mathbf{V} \bullet \mathbf{F} = VF \cos(\arg \mathbf{F} - \arg \mathbf{V}) \qquad (3.41a)$$

Also,

$$\mathbf{V} \bullet \mathbf{F} = V_x F_x + V_y F_y \qquad (3.41b)$$

(For a proof, see Exer. 3.4.)

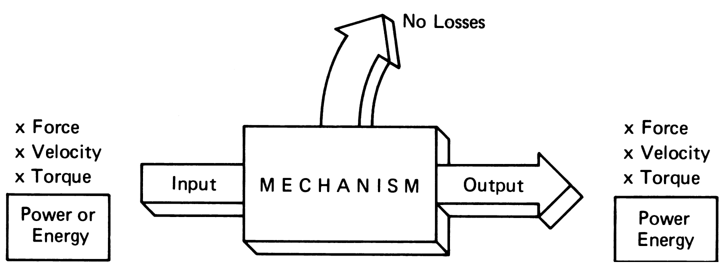


Figure 3.65 Power and energy are conserved through the mechanism. Force, velocity and torque are not.

Figure 3.65 reminds us that neither force, velocity, nor torque alone is constant through a linkage mechanism. The designers of many misguided perpetual motion machines disregarded this fact.

Notice that the units of torque times angular velocity and the scalar product of force and velocity both represent power. From Eq. (3.40),

$$\boxed{\frac{T_{\text{out}}}{T_{\text{in}}} = \frac{\omega_{\text{in}}}{\omega_{\text{out}}}} \tag{3.42}$$

By definition, *mechanical advantage* (M.A.) is the ratio of the magnitudes of *force out* over *force in:*

$$\boxed{\text{M.A.} = \frac{F_{\text{out}}}{F_{\text{in}}}} \tag{3.43}$$

where $F = |\mathbf{F}|$.

Combinings Eqs. (3.43) and (3.40) by noticing that torque is the product of force times a radius,

$$\text{M.A.} = \left(\frac{T_{\text{out}}}{r_{\text{out}}}\right)\left(\frac{r_{\text{in}}}{T_{\text{in}}}\right) = \left(\frac{r_{\text{in}}}{r_{\text{out}}}\right)\left(\frac{T_{\text{out}}}{T_{\text{in}}}\right) \tag{3.44}$$

and

$$\boxed{\text{M.A.} = \left(\frac{r_{\text{in}}}{r_{\text{out}}}\right)\left(\frac{\omega_{\text{in}}}{\omega_{\text{out}}}\right)} \tag{3.45}$$

Thus the mechanical advantage is a product of *two* factors: (1) a ratio of distances that depend on the placement of the input and output forces and (2) an angular velocity ratio. The first factor may not change in value as the mechanism moves, but the second one will change in most linkage mechanisms. Since the angular velocity ratio can be expressed entirely in terms of directed distances (based on the instant-center development), the mechanical advantage can be expressed entirely in terms of ratios of distances (see Sec. 3.8).

Let us look at the four-bar mechanism in Fig. 3.66. If we neglect the weight of links 2, 3, and 4, what reading would you expect the scale to display as a result of the block weighing 10 lbf on link 2 of the mechanism? Using Eq. (3.45),

$$\frac{F_{\text{out}}}{F_{\text{in}}} = \text{M.A.} = \left(\frac{\omega_{\text{in}}}{\omega_{\text{out}}}\right)\left(\frac{r_{\text{in}}}{r_{\text{out}}}\right) \tag{3.46}$$

In this case link 2 is the input while link 4 is the output. Thus,

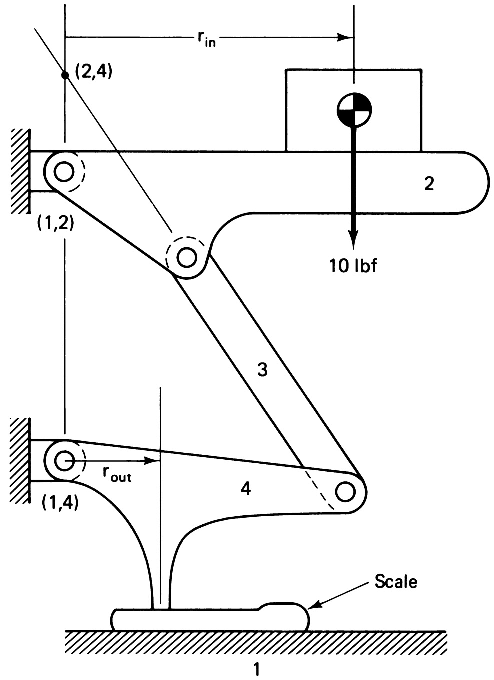


Figure 3.66 Determination of the scale reading based on a weight of 10 lbf.

$$\text{M.A.} = \frac{F_{\text{out}}}{F_{\text{in}}} = \left(\frac{\omega_2}{\omega_4}\right)\left(\frac{r_{\text{in}}}{r_{\text{out}}}\right) = \frac{(1{,}4 - 2{,}4)}{(1{,}2 - 2{,}4)}\,\frac{(r_{\text{in}})}{(r_{\text{out}})} \tag{3.46a}$$

Note that the common instant center (2,4) is outside the others, making the angular velocity ratio positive. Measuring distances on Fig. 3.66* and solving for F_{out},

$$F_{\text{out}} = F_{\text{in}}(\text{M.A.}) = (10)\,\frac{(2)}{(.5)}\,\frac{(1.5)}{(.5)} = (10)\,(4)\,(3) \;=\; 120 \text{ lbf}$$

where M.A. was (4)(3) = 12. The gain in mechanical advantage has contributions both from the radius ratio and the angular velocity ratio. Both are distances measured on the diagram.

This result can be verified by way of free-body diagrams of Fig. 3.67. Here too, the mechanical advantage is purely in terms of distances.

$$\text{M.A.} = \left(\frac{1.5}{0.3}\right)\left(\frac{1.2}{0.5}\right) = (5)\,(2.4) = 12$$

Expressions in the form of Eq. (3.46) are *powerful design tools* and can usually be verified by inspection. In many design situations, the mechanical advantage expression for a mechanism will allow the optimal redesign of that device for improved mechanical advantage. Practical considerations such as the maximum permitted size of the mechanism will usually limit the amount of change allowable from an original design (see Chap. 8 appendix).

*51 mm in Fig. 3.66 = 2 in.

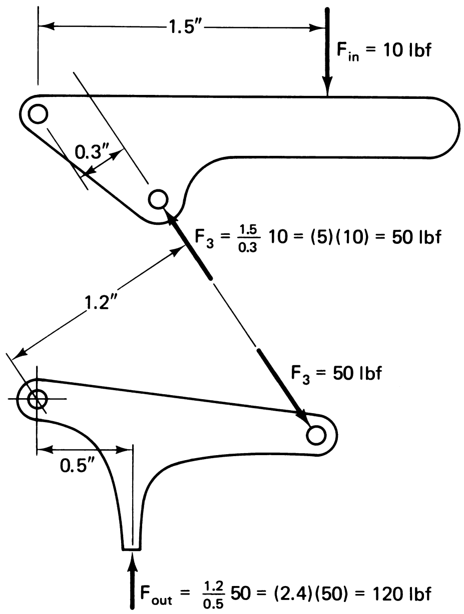


Figure 3.67 Determination of mechanical advantage by way of free-body diagrams.

For example, suppose that the four-bar linkage of Fig. 3.68 is being used as the driving mechanism of a manually operated pump. In the position shown, the handle is being pulled left with force $\mathbf{F}_{in}$. Meanwhile, the pressure difference across the piston in the cylinder is resisting the movement by a force equal and opposite to $\mathbf{F}_{out}$. What is the mechanical advantage of this device in the position shown? (The piston rod is instantaneously perpendicular to link 4.)

If the input link is identified as link 2 and the output link as link 4, then according to Eq. (3.42),

$$\frac{T_4}{T_2} = \frac{\omega_2}{\omega_4}$$

Instant center (2,4) is found to lie between (1,2) and (1,4), so that

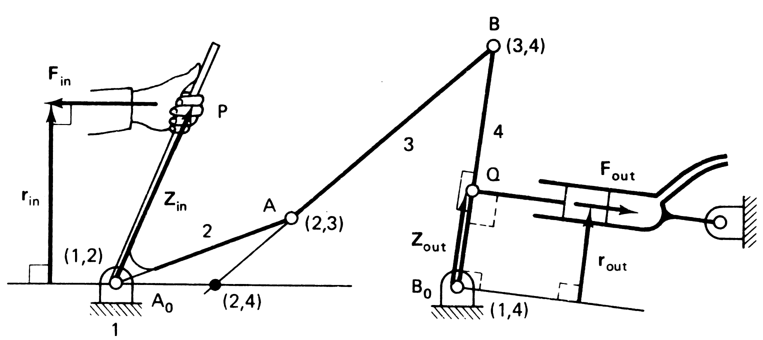


Figure 3.68 Mechanical advantage of this four-bar handpump drive mechanism increases as the toggle position (A_0, A, and B collinear) is approached.

$$\frac{T_4}{T_2} = \frac{\omega_2}{\omega_4} = \frac{(\overrightarrow{1,4 - 2,4})}{(\overrightarrow{1,2 - 2,4})} \tag{3.47}$$

Using complex numbers, Eq. (3.47) will have the form

$$\frac{T_4}{T_2} = \frac{z_{out}F_{out} \sin(\arg \mathbf{F}_{out} - \arg \mathbf{z}_{out})}{z_{in}F_{in} \sin(\arg \mathbf{F}_{in} - \arg \mathbf{z}_{out})}$$

and

$$\text{M.A.} = \frac{F_{out}}{F_{in}} = \frac{z_{in}}{z_{out}} \left(\frac{(\overrightarrow{1,4 - 2,4})}{(\overrightarrow{1,2 - 2,4})}\right) \frac{\sin(\arg \mathbf{F}_{in} - \arg \mathbf{z}_{in})}{\sin(\arg \mathbf{F}_{out} - \arg \mathbf{z}_{out})} \tag{3.48}$$

If we disregard the algebraic sign of the instant-center distance ratio and write r instead of z for the arm lengths of the input and output forces, Eq. (3.48) becomes

$$\text{M.A.} = \left(\frac{r_{in}}{r_{out}}\right) \frac{(1,4 - 2,4)}{(1,2 - 2,4)} \tag{3.49}$$

Notice the factors that make up the mechanical advantage. The mechanical advantage is greater if the ratio z_{in}/z_{out} is greater. This checks intuitively with Fig. 3.68. The second ratio can be checked intuitively also. Notice that as point P is moved to the left (link 2 rotated counterclockwise) to the position shown in Fig. 3.69, where the four-bar linkage is in its "toggle" position, centers (1,2) and (2,4) coincide. According to Eq. (3.48), the mechanical advantage goes to infinity in this position. Since links 2 and 3 line up, (ideally) no force is required at P to resist an infinite force at Q. Of course, bending of link 4 would occur before an infinite force could be applied. The ratio of sines is a measure of the relative closeness to perpendicularity of each force to its arm vector. With these considerations, Eq. (3.48) is intuitively verified.

If more mechanical advantage is required with A_0P in the position shown in Fig. 3.68, then Eq. (3.48) dictates the following possibilities:

1. Increase z_{in}.
2. Decrease z_{out}.
3. Move B_0 away from (2,4) (keeping the rest of the linkage the same).
4. Move A_0 toward (2,4) (keeping the rest of the linkage the same).
5. Move point A until links 2 and 3 line up.
6. Make $\mathbf{F}_{in}$ perpendicular to $\mathbf{z}_{in}$.

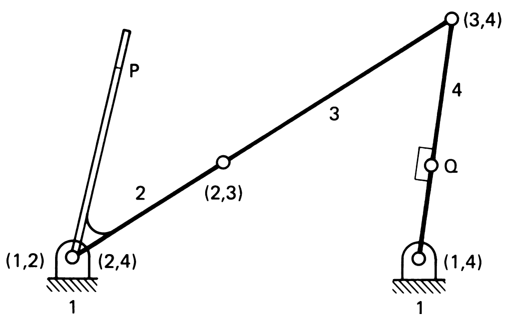


Figure 3.69 Toggle position of the pump-drive mechanism of Fig. 3.68: M.A. is theoretically infinite: a small force at P can overcome a very large force at Q.

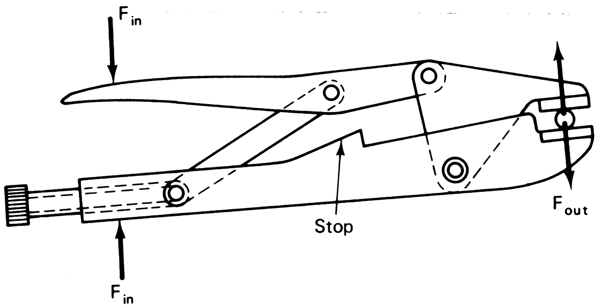


Figure 3.70 The stop prevents excessive overtravel beyond the toggle position of the upper handle and the coupler link.

Example 3.10

Determine the mechanical advantage of the adjustable toggle pliers in Fig. 3.70. Why is this device designed this way?

Solution Let us designate input link = 3, output link = 4, and ground link = 1 (see Fig. 3.71). Thus $T_3\omega_3 = T_4\omega_4$, since $\mathbf{F}_{out}$ is perpendicular to $\mathbf{r}_{out}$ and $\mathbf{F}_{in}$ is perpendicular to $\mathbf{r}_{in}$:

$$T_4 = [\mathbf{r}_{out} \times \mathbf{F}_{out}] = r_{out}F_{out}\sin(\arg \mathbf{F}_{out} - \arg \mathbf{r}_{out}) = r_{out}F_{out}\sin(-90°) = -1.9F_{out}$$

Similarly, $T_3 = [\mathbf{r}_{in} \times \mathbf{F}_{in}] = (5.1)F_{in}$. From Eq. (3.42),

$$\frac{T_4}{T_3} = \frac{\omega_3}{\omega_4} = \frac{\overrightarrow{(3{,}4 - 1{,}4)}}{\overrightarrow{(3{,}4 - 1{,}3)}} = \frac{-1.6}{0.7} = -2.29\ *$$

from which

$$\text{M.A.} = \frac{F_{out}}{F_{in}} = \left|\frac{r_{in}}{r_{out}}\frac{\omega_3}{\omega_4}\right| = \left|\frac{5.1}{1.9}(-2.29)\right| = 6.15^{\dagger}$$

In the position shown, the pliers have only a 6.15:1 mechanical advantage. As we clamp down on the pliers, however, instant center (2,4) approaches center (1,2). When (1,2), (2,3), and (3,4) are close to being in a straight line, (2,4) is nearly coincident with (1,2), (1,3) approaches (3,4), and the mechanical advantage approaches infinity.

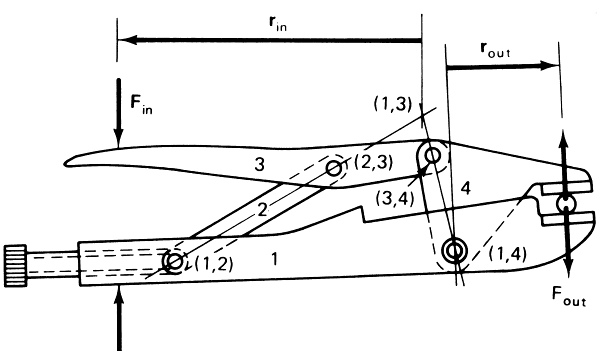


Figure 3.71 As (3,4), (2,3) and (1,2) approach collinearity, M.A. approaches infinity.

*15 mm in Fig. 3.71 = 1.9 units.

†Notice that we could have gone directly to this step, as suggested in Eq. (3.45).

The screw adjustment should be set so that the maximum mechanical advantage occurs at the required distance between the jaws of the pliers. In fact, in some brands, there is a stop located in the “over center” position (just past the dead center), as shown in Fig. 3.70. This gives both a very high mechanical advantage and stable “grip” for the linkage, since it would take an ideally infinite force at the jaws to move the linkage back through its toggle position.

Example 3.11

Determine the mechanical advantage of the slider-crank mechanism shown in Fig. 3.72. The slider output link requires a different approach for the solution.

Solution If link 2 is considered the input link and link 4 the output link, the procedure described previously must be modified (because link 4 is slider so that $\omega_4 = 0$). We know that $(\text{power})_{\text{in}} = (\text{power})_{\text{out}}$. Thus, from Eqs. (3.40) and (3.41),

$$T_2\omega_2 = \mathbf{F}_{\text{out}} \bullet \mathbf{V}_B \tag{3.50}$$

where $T_2 = [\mathbf{z}_{\text{in}} \times \mathbf{F}_{\text{in}}]$. Notice that since the output link 4 is constrained to translate in the horizontal slot, any point considered as part of link 4 must have a velocity in the horizontal direction. Moreover, any point considered as part of link 4 has the same velocity, including the point of the extended plane of link 4 momentarily coincident with the center (2,4). This point has velocity $i\omega_2\overrightarrow{(1,2-2,4)}$; therefore,

$$\mathbf{V}_B = i\omega_2\overrightarrow{(1,2-2,4)} \tag{3.51}$$

Combining (3.50) and (3.51), from (power in) = (power out), we get

$$[\mathbf{z}_{\text{in}} \times \mathbf{F}_{\text{in}}]\omega_2 = T_2\omega_2 = \mathbf{F}_a \bullet \mathbf{V}_B$$

$$\mathbf{F}_a \bullet (i\omega_2\overrightarrow{(1,2-2,4)}) = F_a|\omega_2\overrightarrow{(1,2-2,4)}|\cos(\arg \mathbf{F}_a - \arg \mathbf{V}_B)$$

$$\omega_2 z_{\text{in}} F_{\text{in}} \sin(\arg \mathbf{F}_{\text{in}} - \arg \mathbf{z}_{\text{in}}) = F_a|\omega_2\overrightarrow{(1,2-2,4)}|\cos(\arg \mathbf{F}_a - \arg \mathbf{V}_B)$$

from which, noting that

$$F_a \cos(\arg \mathbf{F}_a - \arg \mathbf{V}_B) = F_{\text{out}} \quad \text{and} \quad \frac{\omega_2}{|\omega_2|} = -1$$

we have

$$\text{M.A.} = \frac{F_{\text{out}}}{F_{\text{in}}} = -\frac{z_{\text{in}} \sin(\arg \mathbf{F}_{\text{in}} - \arg \mathbf{z}_{\text{in}})}{|\overrightarrow{(1,2-2,4)}|} = \frac{r_{\text{in}}}{|\overrightarrow{(1,2-2,4)}|} \tag{3.52}$$

which is positive (as it should be) because $\sin(\arg \mathbf{F}_{\text{in}} - \arg \mathbf{z}_{\text{in}})$ is negative.

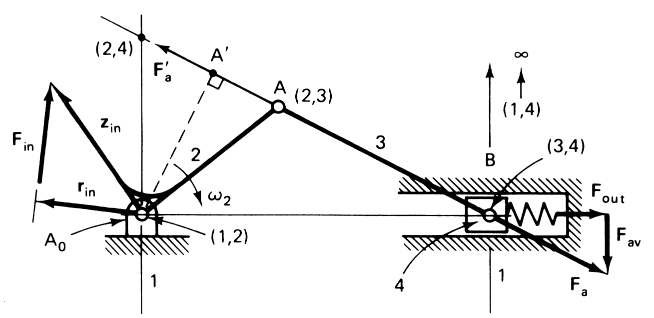


Figure 3.72 Geometric constructions toward finding the M.A. of a slider-crank mechanism.

This expression makes sense: The longer the arm $\mathbf{r}_{\text{in}}$ on the input link, provided that its direction remains the same, the higher the mechanical advantage. Note also that as the input link rotates cw, point A moves toward the toggle position and center (2,4) moves toward (1,2)—increasing the mechanical advantage.

Another verification of Eq. (3.52) follows. Input torque $T_{\text{in}} = [\mathbf{z}_{\text{in}} \times \mathbf{F}_{\text{in}}]$ cw. T_{in} is resisted by the moment of a pin force F'_a at joint A. Since link 3 is a "two-pin" link, F'_a must act along link 3. Its resisting moment is $[(\overrightarrow{(1,2) - A'}) \times \mathbf{F}'_a]$ (ccw), where $(\overrightarrow{(1,2) - A'})$ is perpendicular to link 3. Link 3 transmits $\mathbf{F}'_a$ to slider 4 at (3,4), where it is resisted by $\mathbf{F}_a$. The vertical component $\mathbf{F}_{av}$ of $\mathbf{F}_a$ is perpendicular to the motion of output link 4 and therefore does no work. Thus, the output force $\mathbf{F}_{\text{out}}$ is the horizontal component of $\mathbf{F}_a$, pointing to the right in Fig. 3.72. By similar triangles, it is easy to show that

$$\frac{F_{\text{out}}}{F_a} = \frac{(1{,}2 - \mathrm{A}')}{(1{,}2 - 2{,}4)} \tag{3.53}$$

where, for instance,

$$(1{,}2) - A' = |\overrightarrow{(1{,}2) - A'}|$$

But

$$\frac{F_a}{F_{\text{in}}} = \frac{r_{\text{in}}}{(1{,}2 - A')} \tag{3.54}$$

and by multiplication of these two ratios we find that

$$\frac{F_{\text{out}}}{F_{\text{in}}} = \frac{r_{\text{in}}}{(1{,}2 - 2{,}4)}$$

which agrees with Eq. (3.52).

Notice that link 3 could also be considered as the output link because point B is also a part of the connecting rod. In this case, centers (1,2), (2,3), and (1,3) would be used to obtain an expression for mechanical advantage that would be numerically equivalent to Eq. (3.52) (see Exer. 3.5).

For still another verification of Eq. (3.52), as well as for the verification of the graphical (geometric) construction to determine the mechanical advantage, we resort to free-body diagrams as follows. For the free-body diagram of link 2 (see Fig. 3.73), summing moments about A_0 yields an expression of β_{in} and β_A in terms of the arguments of the vectors involved:

$$\mathbf{Z}_{\text{in}} \times \mathbf{F}_{\text{in}} + \mathbf{r}_2 \times \mathbf{F}'_a = 0$$

$$Z_{\text{in}} F_{\text{in}} \sin \beta_{\text{in}} + r_2 F'_a \sin \beta_A = 0$$

$$Z_{\text{in}} F_{\text{in}} \sin (\arg \mathbf{F}_{\text{in}} - \arg \mathbf{r}_{\text{in}}) + r_2 F'_a \sin (\arg \mathbf{F}'_a - \arg \mathbf{r}_2) = 0$$

Note that β_{in} is cw, and therefore negative, which checks with the fact that the input torque is cw. Also, $z_{\text{in}} \sin \beta_{\text{in}}$ is the perpendicular arm of $\mathbf{F}_{\text{in}}$ about A_0, and similarly, $z_2 \sin \beta_2$ is that of $\mathbf{F}'_a$ about A_0. Solving for F'_a,

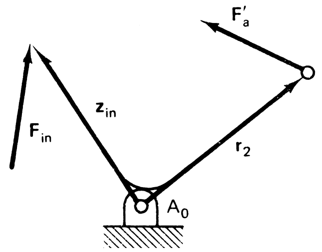


Figure 3.73

$$F'_\alpha = -\frac{z_{\text{in}} F_{\text{in}} \sin \beta_{\text{in}}}{r_2 \sin \beta_2} > 0 \quad \text{because} \quad \frac{\sin \beta_{\text{in}}}{\sin \beta_2} < 0$$

Continuing in this manner, Eq. (3.52) is verified in still another way (see Exer. 3.6).

Example 3.12

A six-link function-generator linkage is shown in Fig. 3.74. (a) Find the location of all the instant centers for this mechanism; (b) if the velocity of point P is known to be 10 m/sec, find ω_3 and $\mathbf{V}_B$ by the instantaneous-center method; (c) if force F_{in} is acting at P (see Fig. 3.74), find F_{out} by instant centers.

Solution (a) Figure 3.75 shows the location of all the instant centers for this mechanism. (b) *Method 1:* Figure 3.76 shows how the instant-center graphical technique can be used to solve for ω_3 and $\mathbf{V}_B$. From this figure,

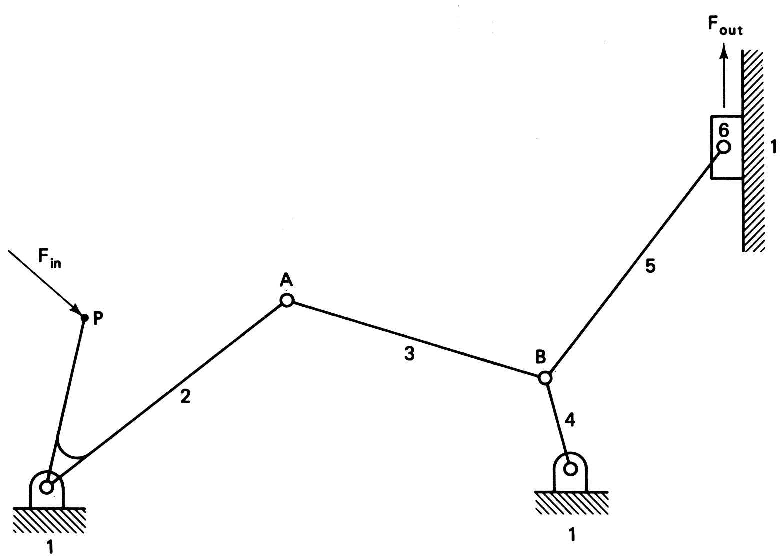


Figure 3.74 Six-link force transmission mechanism.

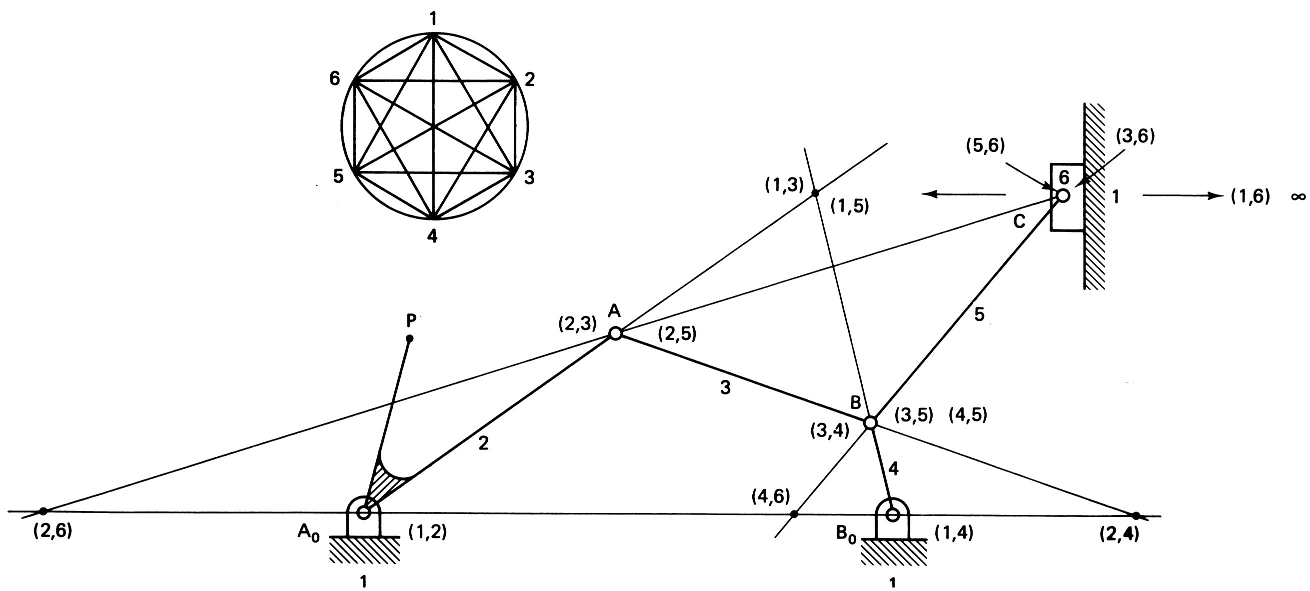


Figure 3.75 Instant centers in six-link mechanism.

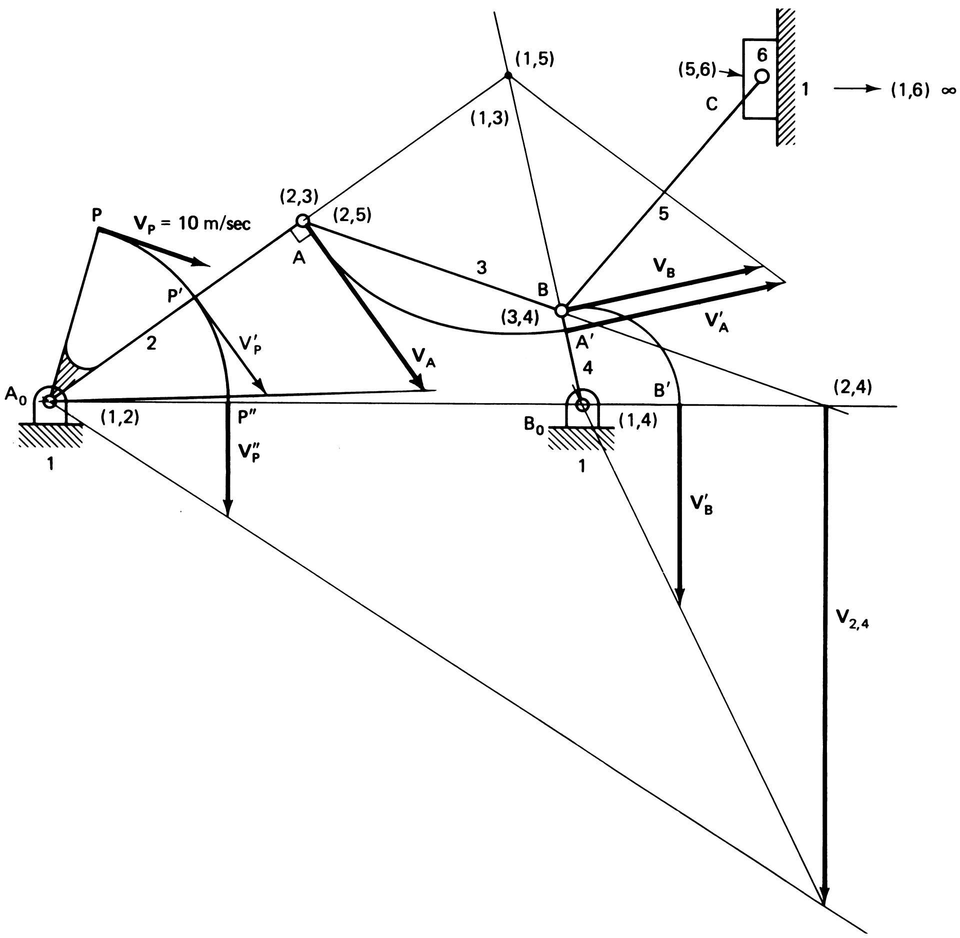


Figure 3.76 Velocity construction using instant centers and gauge lines in six-link mechanism. Scale: 1 mm in Fig. 3.74 = 1.64 mm in the calculations of Example 3.12.

$$V_A = 17.2 \text{ m/sec}$$

$$\omega_3 = \frac{\mathbf{V}_A}{i(\overrightarrow{1{,}3 - 2{,}3})} = \frac{17.2}{0.054} = 318 \text{ rad/sec}$$

$$V_{2,4} = 44.7 \text{ m/sec}$$

$$\mathbf{V}_B = (17.1 \text{ m/sec})i(-e^{i[\arg(\overrightarrow{B_0B})]})$$

Method 2: Using instant-center equations only (Fig. 3.75), since

$$\frac{\omega_3}{\omega_2} = \frac{(\overrightarrow{1{,}2 - 2{,}3})}{(\overrightarrow{1{,}3 - 2{,}3})} = -1.22$$

$$\omega_2 = \frac{\mathbf{V}_P}{i(\overrightarrow{A_0P})} = \frac{-V_P}{A_0P} = -260.8 \text{ rad/sec (cw)}$$

then

$$\omega_3 = \frac{-V_P}{A_0P}(-1.22) = 318.2 \text{ rad/sec (ccw)}$$

$$\omega_4 = \omega_2 \frac{\overrightarrow{(1,2 - 2,4)}}{\overrightarrow{(1,4 - 2,4)}} = (-260.8)(3.10) = -808.0 \text{ rad/sec (cw)}$$

Thus

$$\mathbf{V}_B = i\omega_4(\overrightarrow{B_0B}) = (17.3 \text{ m/sec})i(-e^{i \arg(\overrightarrow{B_0B})})$$

(c) *Method 1:* Using instant centers (1,2), (1,5), and (2,5) (Fig. 3.76),

$$\frac{T_5}{T_2} = \frac{\omega_2}{\omega_5} = \frac{\overrightarrow{(1,5 - 2,5)}}{\overrightarrow{(1,2 - 2,5)}} = -\frac{0.054}{0.066} = -0.820$$

and (Figs. 3.74 and 3.75),

$$T_5 = F_{\text{out}}(0.054); \qquad T_2 = -F_{\text{in}}(0.031)$$

then

$$F_{\text{out}} = \frac{-F_{\text{in}}}{(0.054)}(0.031)(-0.820) = 0.471F_{\text{in}}$$

Method 2: Using instant centers (1,2), (1,6), and (2,6), from Eq. (3.52),

$$F_{\text{out}} = \left|\frac{T_2}{(1,2 - 6,2)}\right|$$

then

$$F_{\text{out}} = \left|\frac{-F_{\text{in}}(0.031)}{0.057}\right| = 0.54F_{\text{in}}$$

3.10 ANALYTICAL METHOD FOR VELOCITY AND MECHANICAL ADVANTAGE DETERMINATION

The procedures described previously for velocity and mechanical advantage analysis are basically graphical solution procedures. Some analytical, complex-number equivalents have been presented. If only a finite number of positions of a linkage are to be analyzed, any of the graphical procedures are certainly warranted. If, however, a large number of positions and/or a large number of linkages need to be analyzed, the graphical procedures are too time consuming. With micro, desktop, or laptop computers or programmable hand calculators readily available, the analytical expressions based on complex-number representation are extremely valuable. After one becomes comfortable with the graphically based procedures (thereby gaining clear visualization of vector solutions), purely analytical methods can be used with greater confidence. When a question arises from the nonvisual techniques, a graphical spot check can verify the results. Also, computer graphics can be used to display visually the results of analytical methods (see Sec. 3.11, the Web examples created for this book, and the color inserts in this book).

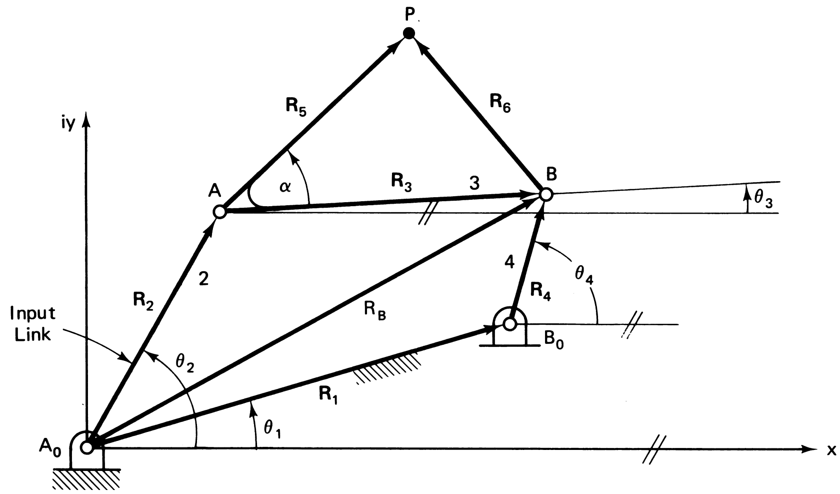


Figure 3.77 Vector notation for velocity analysis by complex numbers in four-bar mechanism.

Referring to Fig. 3.77, which shows a vector representation of a four-bar linkage, the angular velocities of links 3 and 4 may be determined as functions of the input link (link 2) angular velocity and the displacement position parameters. Recall that a displacement analysis for this same linkage was presented in Sec. 3.3.

The position vector from the base of the input link (A_0) to point B may be written via two routes:

$$\mathbf{R}_B = \mathbf{R}_2 + \mathbf{R}_3 \tag{3.55}$$

$$\mathbf{R}_B = \mathbf{R}_1 + \mathbf{R}_4 \tag{3.56}$$

The derivatives of these expressions will yield the velocity of point B. Using the polar form of these expressions for taking time derivatives leads to

$$\mathbf{R}_B = r_2 e^{i\theta_2} + r_3{}^{i\theta_3} = r_1 e^{i\theta_1} + r_4 e^{i\theta_4} \tag{3.57}$$

$$\mathbf{V}_B = r_2\omega_2 i e^{i\theta_2} + r_3\omega_3 i e^{i\theta_3} = r_4\omega_4 i e^{i\theta_4} \tag{3.58}$$

Now using the Cartesian form of Eq. (3.58) to break it into real and imaginary parts, we obtain

$$\begin{aligned} r_2\omega_2 \sin\theta_2 + r_3\omega_3 \sin\theta_3 &= r_4\omega_4 \sin\theta_4 \\ r_2\omega_2 \cos\theta_2 + r_3\omega_3 \cos\theta_3 &= r_4\omega_4 \cos\theta_4 \end{aligned} \tag{3.59}$$

These two scalar equations contain only two scalar unknowns, ω_3 and ω_4. Multiplying the first equation by $\cos\theta_4$ and the second by $\sin\theta_4$, we have

$$\begin{aligned} r_2\omega_2 \sin\theta_2 \cos\theta_4 + r_3\omega_3 \sin\theta_3 \cos\theta_4 &= r_4\omega_4\ \theta_4 \cos\theta_4 \sin\theta_4 \\ r_2\omega_2 \cos\theta_2 \sin\theta_4 + r_3\omega_3 \cos\theta_3 \sin\theta_4 &= r_4\omega_4\ \theta_4 \sin\theta_4 \cos\theta_4 \end{aligned} \tag{3.60}$$

Subtracting the first from the second eliminated ω_4:

$$r_2\omega_2(\cos\theta_2 \sin\theta_4 - \sin\theta_2 \cos\theta_4) + r_3\omega_3(\cos\theta_3 \sin\theta_4 - \sin\theta_3 \cos\theta_4) = 0$$

or

$$r_2\omega_2 \sin(\theta_4 - \theta_2) + r_3\omega_3 \sin(\theta_4 - \theta_3) = 0 \tag{3.61}$$

Thus

$$\boxed{\omega_3 = -\frac{r_2}{r_3}\omega_2\frac{\sin(\theta_4 - \theta_2)}{\sin(\theta_4 - \theta_3)}} \tag{3.62}$$

Eliminating the terms containing ω_3 instead of ω_4 from eq. (3.59) yields

$$\boxed{\omega_4 = \frac{r_2}{r_4}\omega_2\frac{\sin(\theta_3 - \theta_2)}{\sin(\theta_3 - \theta_4)}} \tag{3.63}$$

These two expressions are easily programmed for automatic computation. Notice that Eqs. (3.62) and (3.63) may also be utilized for mechanical advantage analysis, where ω_3/ω_2 are utilized in the M.A. expressions.

Example 3.13

Determine the analytical expression for the velocity of point P of Fig. 3.77.

Solution The position and velocity of point P are expressed in vector form as

$$\mathbf{R}_P = r_2e^{i\theta_2} + r_5e^{i(\theta_3+\alpha)} \tag{3.64}$$

The derivative of this expression yields

$$\mathbf{V}_P = r_2\omega_2ie^{i\theta_2} + r_5\omega_3ie^{i(\theta_3+\alpha)} \tag{3.65}$$

or in Cartesian form

$$\begin{aligned} &\text{Real part:} && V_{Px} = -r_2\omega_2 \sin\theta_2 - r_5\omega_3 \sin(\theta_3 + \alpha) \\ &\text{Imaginary part:} && V_{Py} = r_2\omega_2 \cos\theta_2 + r_5\omega_3 \cos(\theta_3 + \alpha) \end{aligned} \tag{3.66}$$

where ω_3 is found by Eq. (3.62).

Correlation of Mechanical Advantage and Transmission Angle

In Sec. 3.1 we observed that the transmission angle is a means of determining the effectiveness with which motion is imparted to an output link of a particular mechanism. Mechanical advantage was defined as the instantaneous magnitude ratio of the output force

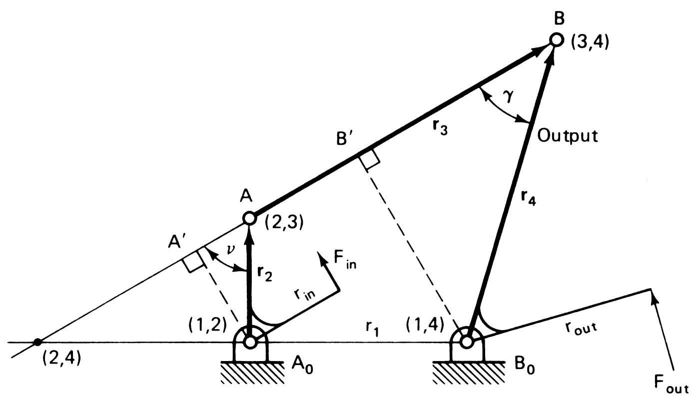


Figure 3.78 Correlation of M.A. with transmission angle γ and crank-coupler angle v in a four-bar mechanism. See Eq. (3.69).

to the input force of a particular mechanism (Sec. 3.9). Both of these indices of performance are static parameters (the effect of inertia is not included) that help us compare one linkage with another, or one position of a linkage with another position of the same linkage. Both of these indices can be expressed as a function of the geometry of the linkage. A comparison is warranted to avoid confusion of these terms.

In Fig. 3.78 the mechanical advantage may be expressed as [Eq. (3.46a)]

$$\text{M.A.} = \frac{F_{\text{out}}}{F_{\text{in}}} = \left(\frac{r_{\text{in}}}{r_{\text{out}}}\right)\left|\frac{\omega_2}{\omega_4}\right|$$

$$= \frac{r_{\text{in}}}{r_{\text{out}}}\frac{(1{,}4 - 2{,}4)}{(1{,}2 - 2{,}4)}$$

Constructing lines A_0A' and B_0B' perpendicular to the line containing (2,4), A, and B shows that by similar triangles,

$$\frac{(2{,}4 - 1{,}4)}{(1{,}4 - B')} = \frac{(2{,}4 - 1{,}2)}{(1{,}2 - A')} \tag{3.67}$$

If λ and v are the magnitudes of the smaller of the two angles made by the coupler (or its extension) with the output and input links, respectively, then $(1{,}4 - B') = r_4 \sin \lambda$, $(1{,}2 - A') = r_2 \sin \nu$, and

$$\frac{(2{,}4 - 1{,}4)}{(2{,}4 - 1{,}2)} = \frac{r_4 \sin \gamma}{r_2 \sin \nu} \tag{3.68}$$

Thus

$$\text{M.A.} = \frac{r_{\text{in}}}{r_{\text{out}}}\left(\frac{r_4 \sin \gamma}{r_2 \sin \nu}\right) \tag{3.69}$$

Notice that the mechanical advantage becomes infinite when angle ν is either 0° or 180°. This agrees with the analysis of Sec. 3.9 since in either of these cases instant center (2,4) is coincident with center (1,2). Notice also the relationship between the transmission

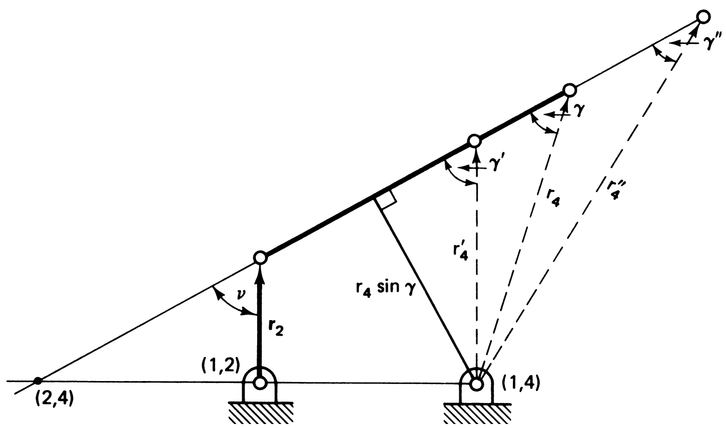


Figure 3.79 If (2,4), (1,2), (1,4), and $\mathbf{r}_2$ are unchanged, variation of transmission angle does not change the M.A.

angle γ and the mechanical advantage (disregarding the ratio r_{in}/r_{out}): If angle γ is 0° or 180°, the mechanical advantage is zero. For other values of mechanical advantage, the transmission angle will vary. In Fig. 3.79, let (2,4 − 1,4), (2,4 − 1,2), r_2, and ν be fixed. The transmission angle γ and the length of the output link r_4 may have different values for a given value of M.A. as long as the product ($r_4 \sin\gamma$) remains constant. The same amount of torque will be transferred to the output link in each of the cases shown in Fig. 3.79, but for the cases with smaller transmission angles, a large component of the static force transmitted through the coupler will result in a larger bearing force at (1,4) rather than in a usable force perpendicular to the output link. Thus a linkage with good mechanical advantage may have an unacceptable transmission angle and a linkage with an excellent transmission angle in a particular position may not have a sufficient mechanical advantage. Since both the transmission angle and mechanical advantage vary with linkage position, either parameter can be critical to the designer in certain positions.

Minimum Value of Mechanical Advantage

Section 3.9 showed that the mechanical advantage becomes infinitely large when the input link and the coupler are in line. Another useful design parameter would be the position in which a linkage attains the minimum value of mechanical advantage. *Freudenstein's theorem* [74] provides a method, expressible in terms of linkage geometry, for predicting the extreme value of angular velocity ratio ω_{out}/ω_{in}, which is the inverse of one of the factors in the mechanical advantage equation [Eq. (3.45)]. The therorem utilizes the two moving instant centers (2,4) and (1,3) shown in Fig. 3.80. The line between these centers is called the *collineation axis.* Freudenstein's theorem states that *at an extreme value of the velocity ratio in a four-bar linkage, the collineation axis is perpendicular to the connecting link AB.* In Fig. 3.80,

$$\left|\frac{\omega_4}{\omega_2}\right| = \frac{(2{,}4 - 1{,}2)}{(2{,}4 - 1{,}4)} = \frac{(2{,}4 - 1{,}2)}{(2{,}4 - 1{,}2) + (1{,}2 - 1{,}4)}$$

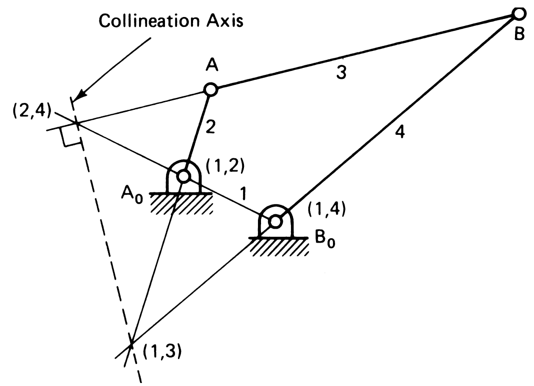


Figure 3.80 Freudenstein's theorem: At extreme of ω_4/ω_2 the collineation axis is perpendicular to coupler 3.

Since (1,2 – 1,4) remains fixed as the linkage moves, the extreme values of the angular velocity ratio occur when the distance (2,4 – 1,2) reaches an extreme value. These positions may occur when the instant center (2,4) is on either side of instant center (1,2). On the other hand, recall that the mechanical advantage is maximum when (2,4 – 1,2) is minimum. During the motion of the linkage, instant center (2,4) moves along the line of centers (1,2) and (1,4). At an extreme value of the mechanical advantage, instant center (2,4) must come instantaneously to rest. This occurs when the velocity of (2,4), considered as part of link 3, is directed along *AB*. This will be true only when link 3 [extended to include (2,4)] is perpendicular to the collineation axis because center (1,3) is the instantaneous center of rotation of link 3. An inversion of this theorem is given by Shigley [148]: An extreme value of the velocity ratio ω_3/ω_2 of the four-bar linkage occurs when the collineation axis is perpendicular to the driven link (link 4).

Limits of Motion of a Four-bar Linkage

Often it is desirable to determine the angular limits of motion of a four-bar linkage. For example, a Grashof rocker-rocker motion or path-generator mechanism may be caused to rock back and forth if a dyad is added to the four-bar as input. This "driving dyad" would have to form (together with the four-bar input link) a crank-rocker, such that it would drive the original four-bar between its limits of motion. Table 3.2* presents the equations governing the extreme limits of angular motion of the input link for the Grashof and non-Grashof four-bars (Sec. 3.1).

3.50. What force ($\mathbf{F}_{in}$) is required at the piston in the mechanism of Fig. P3.58 to balance the weight **W** on link 5?

(a) Use the instant-center method.

(b) Check your result by the complex-number method.

$AB = 4$ m
$AC = 3$ m
$CB = 2.8$ m
$C_0B_0 = 2.9$ m
$O_2C_0 = 3.7$ m
$O_2B_0 = 6.2$ m
$C_0C = 1.5$ m
$BD = 1.5$ m
$B_0D = 1.5$ m
$B_0B = 2.1$ m

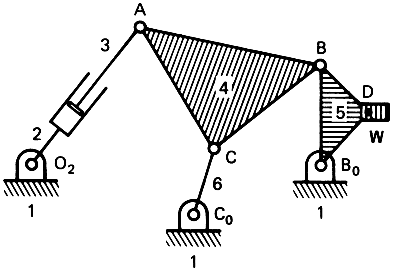


Figure P3.58

3.51. (a) What is the relationship between the input force $\mathbf{F}_{\text{in}}$ and the resisting force $\mathbf{R}$ in Fig. P3.59? Use instant centers.

(b) What is ω_5/ω_2?

(c) Check your results in parts (a) and (b) by the complex-number method.

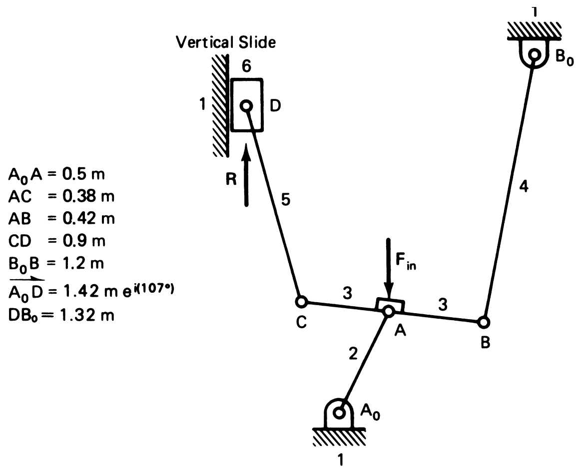


Figure P3.59

3.52. (a) What is the relationship between the input torque T_{in} and the resistance of the cutting tool $\mathbf{R}$ in the shaper mechanism of Fig. P3.60?

(b) What is ω_2/V_6?

(c) Check your results in parts (a) and (b) by the complex-number method.

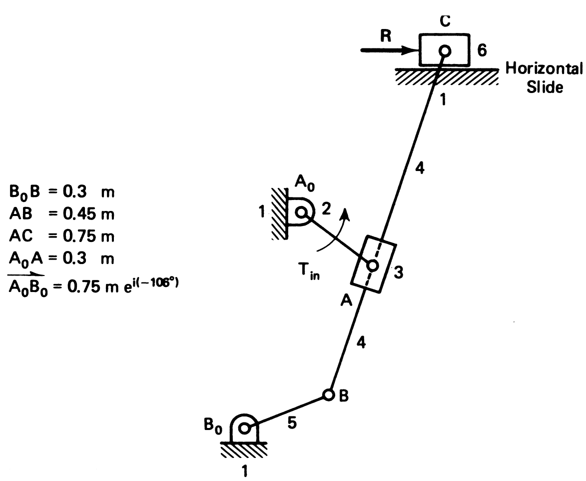


Figure P3.60

3.53. Determine the mechanical advantage ($F_{\text{out}}/F_{\text{in}}$) in the position shown of the inverted slider-crank in Fig. P3.61.

(a) Use instant centers.

(b) If friction exists in the slider generating a resisting force $\mathbf{F}_r$, what is the mechanical advantage of the linkage?

(c) Check your results in parts (a) and (b) the the complex-number method. (Scale the figure for needed data.)

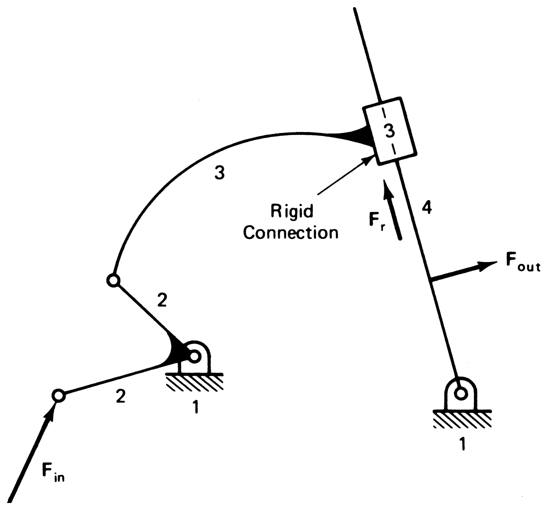


Figure P3.61

3.54. The linkage used in Prob. 3.9 is redrawn in Fig. P3.62; ω_2 is 2 rad/sec cw.

(a) Find all instant-center locations.

(b) Determine the mechanical advantage. (Scale the drawing for the location of $\mathbf{F}_{in}$ and $\mathbf{F}_{out}$. Note that $\mathbf{F}_{out}$ is a resistance force.)

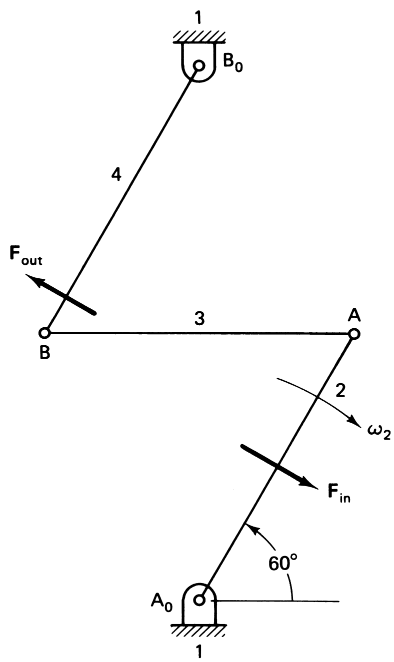


Figure P3.62

3.55. (a) Find the angular velocity ration (ω_4/ω_6) in the position shown of the mobile lifting mechanism in Fig. P3.63.

(b) Find the mechanical advantage of this linkage.

(c) Check your results in parts (a) and (b) by the complex-number method. (Scale the figure for needed data.)

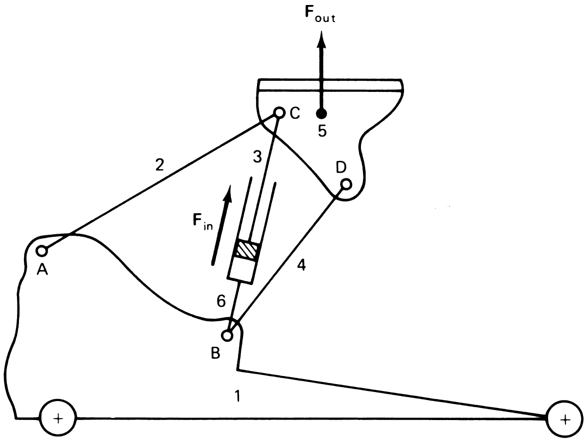


Figure P3.63

3.56. The input link of the Stephenson III six-bar shown in Fig. P3.64 has angular velocity $\omega_2 = 5$ rad/sec cw. Find the mechanical advantage if $\mathbf{F}_{in}$ acts on link 3 and $\mathbf{F}_{out}$ acts on link 6 as shown. Let A_0A be 3″ for scaling purposes; AB is at +30°.

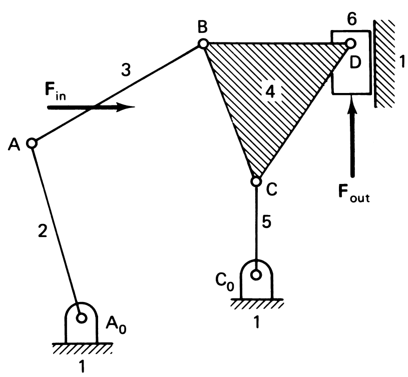


Figure P3.64

3.57. Determine the mechanical advantage of the dumping mechanism shown in Fig. P3.65. The hydraulic cylinder (2) provides the input force ($\mathbf{F}_{in}$). The payload is ($\mathbf{W}$).

(a) Use the method of your choice.

(b) Check your results with another method. (Scale the figure for needed data.)

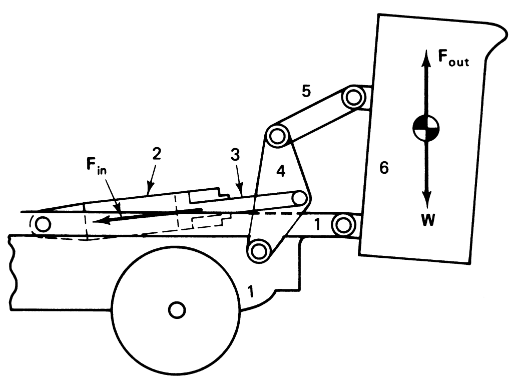


Figure P3.65

3.58. The velocity of slider 2 of the hydraulic actuator linkage is 10 in./sec to the right shown in Fig. P3.66. Link lengths are $AB_0 = 5''$, $BB_0 = 6''$.

(a) Find the angular velocity of link 4.

(b) Determine the mechanical advantage. (Scale the figure for needed data.)

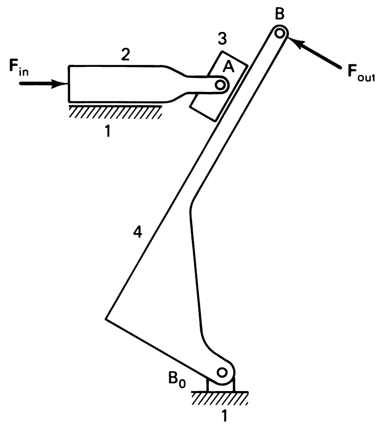


Figure P3.66

3.59. A cam surface is driving a four-bar linkage whose movement is resisted by a spring at point D (Fig. P3.67).

(a) What is the mechanical advantage of this mechanism?

(b) What is ω_5/ω_2?

(c) What is ω_5/ω_3?

(d) Check your results in parts (a) and (b) by the complex-number method. (Scale the figure for needed data.)

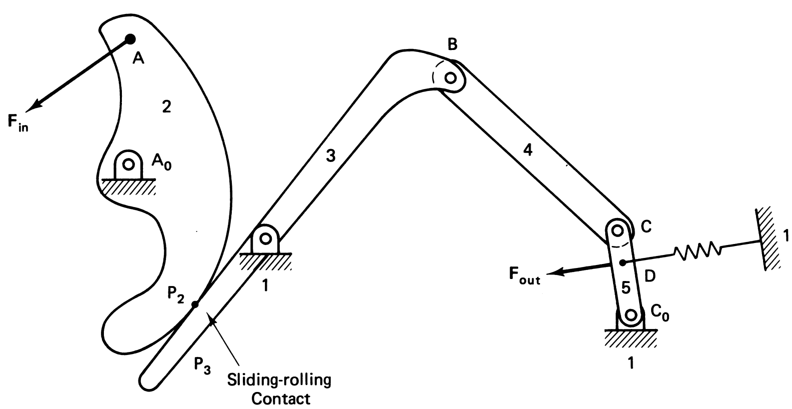


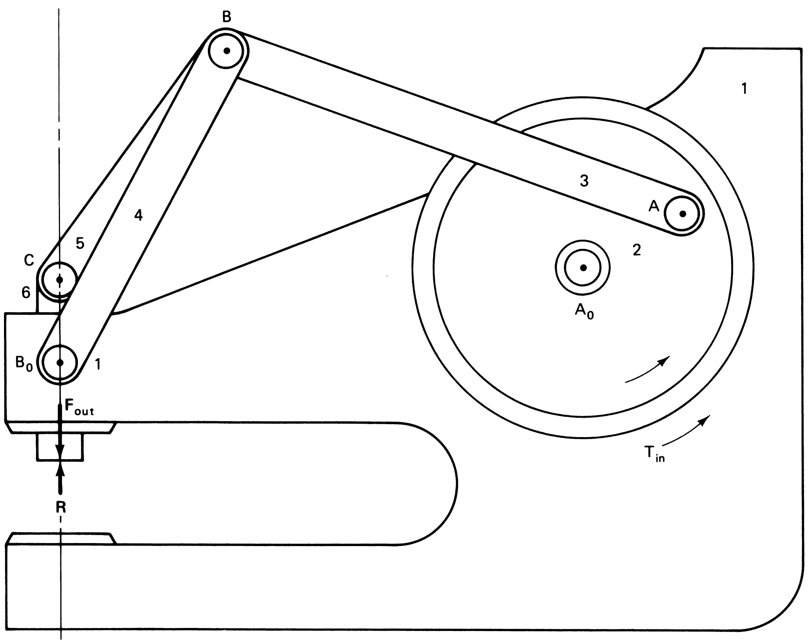
Figure P3.67

3.60. (a) Find the ratio of F_{out}/T_{in} for the riveting mechanisms in both positions shown in Fig. P3.68.

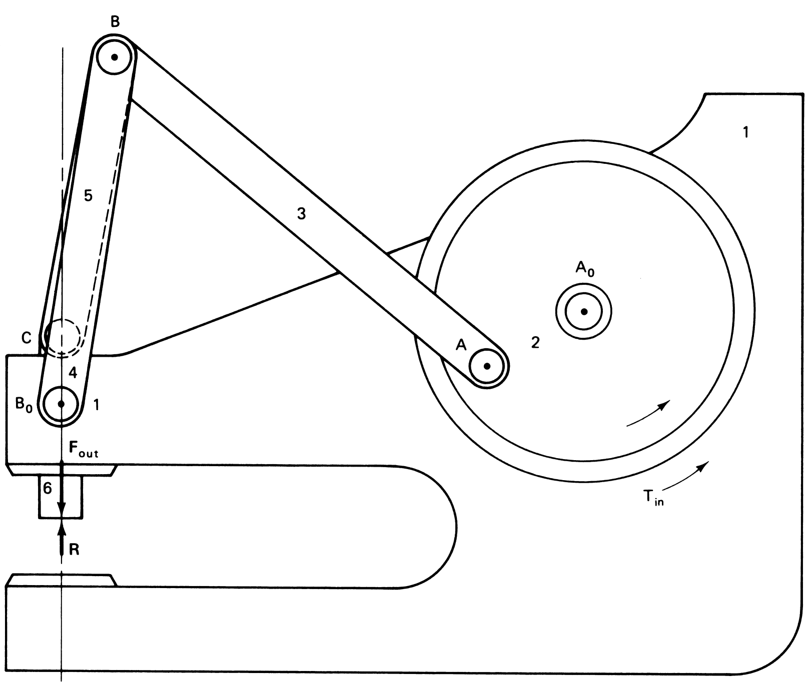
(b) Which position has the greater mechanical advantage?

(c) What is V_6/ω_2 in each case?

(d) Check your results in parts (a) and (b) by the complex-number method. (Scale the figure for needed data.)



(a)



(b)

Figure P3.68

3.61. In the rock-crusher linkage of Fig. P3.69, determine the ratio of F_{out}/T_{in} by two methods:
(a) Use instant centers (1,2), (1,6), and (2,6).
(b) Use instant centers (1,2), (1,4), (2,4), (1,6), and (4,6) (using superposition).
(c) Check your results in parts (a) and (b) by the complex-number method. (Scale the figure for needed data.)

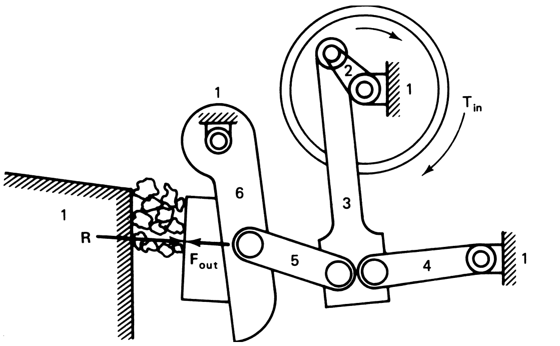


Figure P3.69

3.62. **(a)** Find the mechanical advantage of the straight-line clamp of Fig. P3.70.
(b) What is V_4/ω_2?
(c) Check your results in parts (a) and (b) by the complex-number method. (Scale the figure for needed data.)

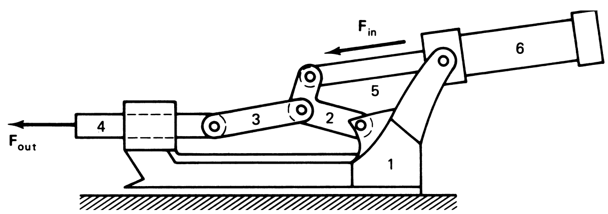


Figure P3.70

3.63. A hold-down clamp is shown in two positions in Fig. P3.71.
(a) Find the mechanical advantage in both the solid and dashed positions.
(b) Check your results by the complex-number method. (Scale the figure for needed data.)

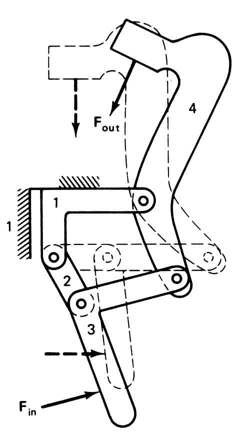


Figure P3.71

3.64. A four-bar linkage has been synthesized to remove the cap from a beverage bottle (see Fig. P3.72).

(a) What is the ratio of the torque applied to the bottle cap to the input force ($\mathbf{F}_{in}$) in the two positions shown?

(b) Determine ω_3/ω_2 in the positions shown.

(c) Check your results in parts (a) and (b) by the complex-number method. (Scale the figure for needed data.)

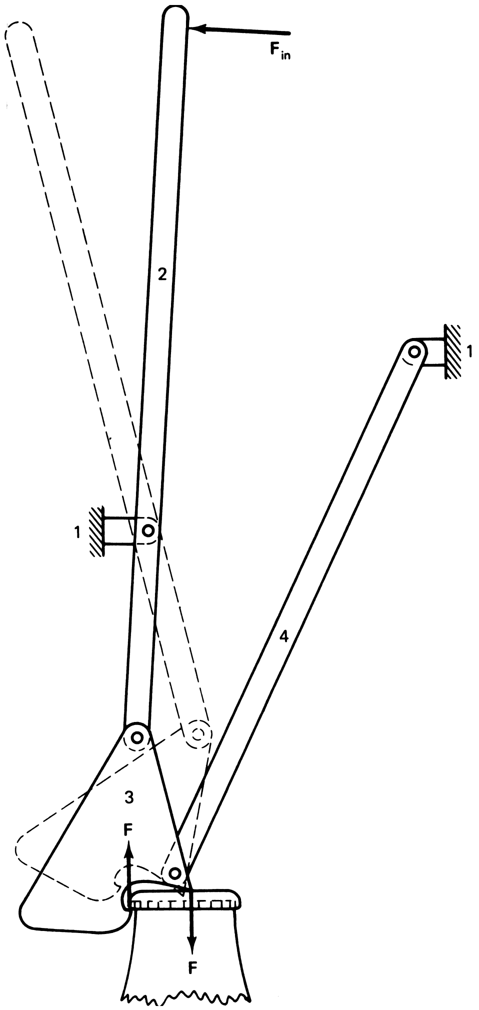


Figure P3.72

3.65. After several unsuccessful attempts to open a can of liquid refreshment, a linkage was designed to assist in the task (Fig. P3.73). Determine the mechanical advantage of this "thirst-while" mechanism.

(a) Use instant centers (2,5), (1,2), and (1,5).

(b) Use instant centers (2,6), (1,2), and (1,6).

(c) Use instant centers (1,2), (2,4), (1,4), and (4,6).

(d) Use all of the foregoing.

(e) Use a can opener.

(f) Check all the foregoing by the complex-number method [except part (e)]. (Scale the figure for needed data.)

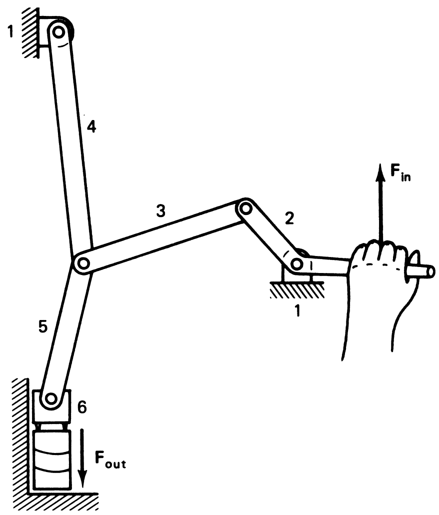


Figure P3.73

3.66. The Watt I six-bar linkage shown in Fig. P3.74 is designed for use as a parallel motion supermarket-based hand-operated can crusher to spur recycling of materials. Such machines require as large a mechanical advantage as possible to amplify the crush force developed from a limited human input. Determine the mechanical advantage of this linkage in the position shown when the crush plate first contacts the can. The handle is being rotated at 1 rad/sec cw. Positional information is $A_0A = 4''$, $A_0D = 6.13''$, $AB = 5.2''$, $AC = 7''$, $BC = 7.97''$, $BB_0 = 5''$, $A_0B_0 = 13''$, $CE = 17.52''$, $DE = 6.74''$, $A_0\mathbf{F}_{in} = 12''$, $E\mathbf{F}_{out} = 24''$; A_0A is at +80°, CE is at −25°, the handle is at +40°.

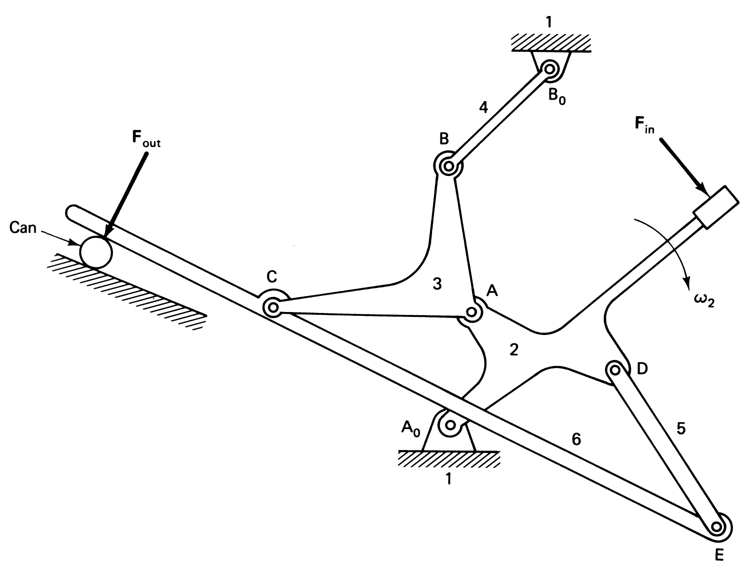


Figure P3.74

3.67. **(a)** What is the ratio of F_{out}/F_{in} of Fig. P3.75? Use instant centers.
(b) If the coefficient of friction in both sliders (not the fork joint) is $\mu = 0.2$, determine the new mechanical advantage.
(c) Check your results in parts (a) and (b) by the complex-number method. (Scale the figure for needed data.)

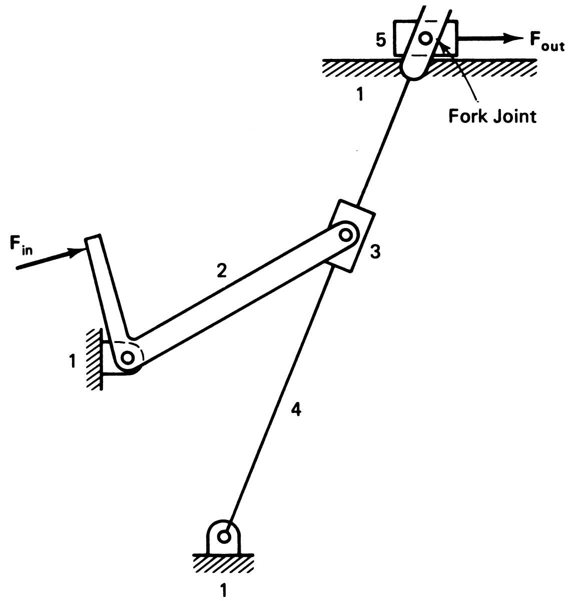


Figure 3.75

3.68. **(a)** Find the angular velocity ratio (ω_4/ω_6) of the mechanism in Fig. P3.76.
(b) Determine the mechanical advantage (F_{out}/F_{in}).
(c) Check your results in parts (a) and (b) by the complex-number method. (Scale the figure for needed data.)

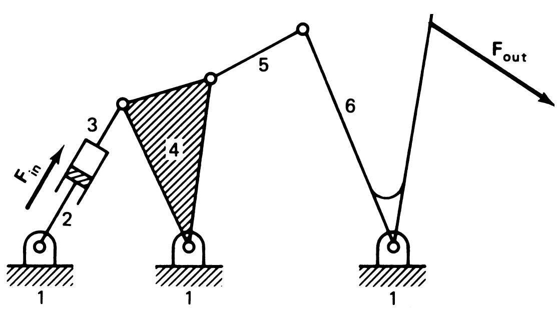


Figure P3.76

3.69. Links 1, 2, 3, and 4 have been designed for the Watt II six-bar shown in Fig. P3.77. It is desired to have link 6 perpendicular to the ground and $\mathbf{F}_{out}$ parallel and 20″ from the ground when the linkage is in the position shown. Link 5 is not yet designed and is shown in its conceptual coupler position only.* If the overall mechanical advantage of the linkage is to be 3, how long should link 6 be? Known linkage dimensions are $A_0B_0 = A_0A = 36.334''$, $AB_0 = BB_0 = 20.881''$, $AB = B_0C_0 = 12''$.

*It will connect link 4 in this current position to link 6 in its final position spanning points B and C.

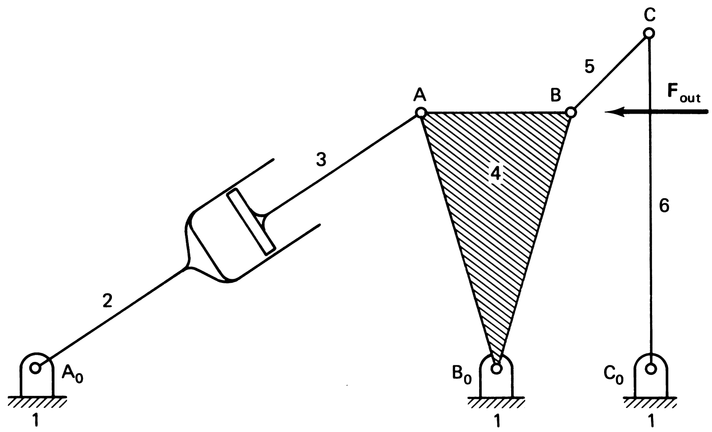


Figure P3.77

3.70. A typical automobile hood linkage is shown in Fig. P3.78. Notice the placement of the counterbalancing spring.

(a) Determine (in terms of **W**) the force exerted by the spring (if other links are assumed to have negligible weight as compared with the hood) to balance **W.**

(b) Does the effectiveness of the spring improve if its attachment point (P) is moved up vertically? Why or why not?

(c) Does the placement of the spring make sense keeping in mind the entire range of motion of the mechanism? Why or why not?

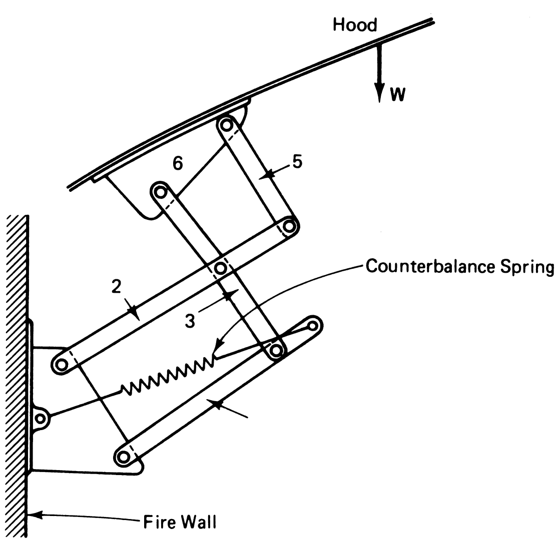


Figure P3.78

3.71. A standard toggle press is shown in Fig. P3.79. Determine its mechanical advantage (force of the plunger divided by the force of the cylinder). Scale the drawing.

3.72. Polycentric motion is said to occur when the center of rotation of a moving body appears to move in plane space. This type of motion behavior is desirable for design situations in which it is desired to have a continuously changing mechanical advantage or in which it is desired

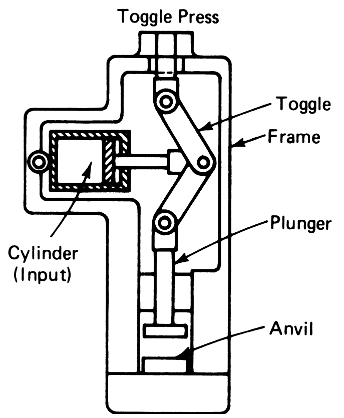


Figure P3.79

to have self-locking behavior. For example [150], the polycentric hinge shown in Fig. P3.80 has a shifting instant center.

(a) Regarding the door as the input, which inversion of the slider-crank is this linkage—similar to Fig. 3.8, 3.9, 3.10, or 3.11?

(b) Plot the trajectory of the instant center of the door with respect to the frame as the door opens.

(c) Describe in practical terms the advantage of these moving instant centers.

(d) Spot-check your results for part (b) with complex numbers. (Scale the figure for needed data.)

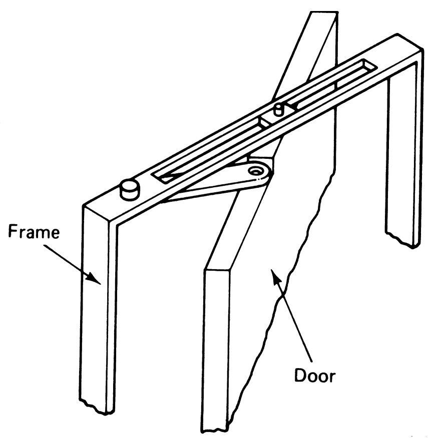


Figure P3.80

3.73. Referring to the casement window linkage of Fig. 1.4a,

(a) Find the locations of the instant centers.

(b) If T_2 is applied to link 2, how much torque (T_4) results at the window?

(c) List the changes in linkage geometry that would improve part (b).

(d) If a frictional force exists at the shoe (due to the weight of the window and a nonzero friction coefficient), find the transmission angle considering the sash (link 4) as the output.

(e) List the changes in linkage geometry that would improve part (d).

(f) What trade-offs in linkage performance will have to be made to improve both parts (b) and (c)?

(g) Spot-check part (a) by complex numbers. (Scale the figure for needed data.)

3.74. Answer the same questions as in Prob. P3.73, but refer to Fig. 1.4b.

3.75. Figure P3.81 shows a Rongeur, which is used by orthopedic surgeons for cutting away bone. The leaf-type springs between the handles return the linkage to the open position so that the Rongeur can be operated by one hand.

(a) What type of linkage is this?

(b) Determine the mechanical advantage of this linkage in the position shown as well as the closed position (disregard the spring). (Scale the figure for needed data.)

(c) Why is this device designed this way?

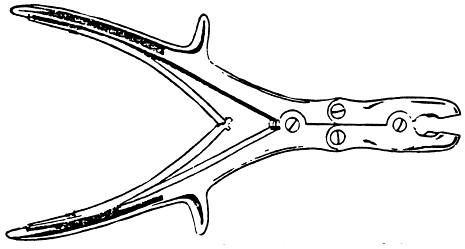


Figure P3.81

3.76. A pair of squeeze clamps are shown in Fig. P3.82.

(a) What type of linkage is this?

(b) Determine the mechanical advantage of this linkage in the position shown as well as in a position such that the handles have rotated toward each other 10° each. (Scale the figure for needed data.)

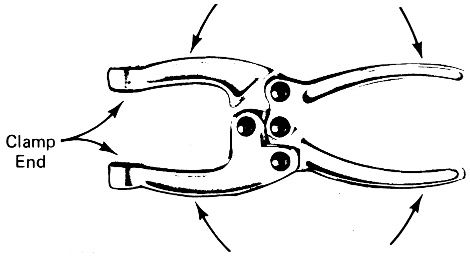


Figure P3.82

3.77. Figure P3.83 shows a mechanical clamp used on machine tools for work-holding fixtures. Determine the mechanical advantage of this linkage. Why is it designed this way? (Scale the figure for needed data.)

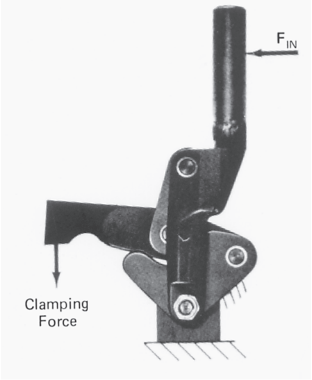


Figure P3.83

3.78. Figure P1.77 is a movable storage bin mechanism that was designed by undergraduate students at the University of Minnesota. Problem 1.46 describes the objectives of this design. The pivot icon with the dot in the middle is the input pivot.
- **(a)** Determine the transmission and deviation angles for each position.
- **(b)** Identify those positions in which the mechanism has poor transmission characteristics.
- **(c)** Can you suggest any design changes to improve these poor characteristics?
- **(d)** Find the location of the instant centers for each design.
- **(e)** If the storage bin weigh 50 lbf and the mass of the mechanism links is considered very small compared to the bin, what is the required input torque at each position?

3.79. Figure P1.78 is a dust pan mechanism that was designed by undergraduate students at the University of Minnesota. Problem 1.46 describes the objectives of this design. The pivot icon with the dot in the middle is the input pivot.
- **(a)** Determine the transmission and deviation angles for each position.
- **(b)** Identify those positions in which the mechanism has poor transmission characteristics.
- **(c)** Can you suggest any design changes to improve these poor characteristics?
- **(d)** Find the location of the instant centers for each design.
- **(e)** If the dust pan plus contents weighs 15 lbf and the mass of the mechanism links is considered very small compared to the pan, what is the required input torque at each position?

3.80. A potential design for bicycle chain removal tool is shown in Fig. P1.82.
- **(a)** For this design, find the locations of all the instant centers.
- **(b)** What is the ratio of input torque to output force (on the chain link pin) in the position shown?

3.81. Figure P1.76 is a monitor mover design that was designed by undergraduate students at the University of Minnesota. The objectives of this design include rotating a computer monitor from a storage position inside a desk to a viewing position. The pivot icon with the dot in the middle is the input pivot.
- **(a)** Determine the transmission and deviation angles for each position.
- **(b)** Identify those positions in which the mechanism has poor transmission characteristics.
- **(c)** Can you suggest any design changes to improve these poor characteristics?
- **(d)** Find the location of the instant centers for each design.
- **(e)** If the monitor weighs 20 lbf and the mass of the mechanism links is considered very small compared to the monitor, what is the required input torque at each position in (b)?

3.82. A potential design for a bicycle chain removal tool is shown in Fig. P1.83.
- **(a)** For the design shown, discuss how you might find the mechanical advantage of this device. How would you model the rachet and pin-in-slot? Draw a unscaled kinematic diagram of this model.
- **(b)** How might you determine the mechanical advantage of this device if you knew the angular rotation of the handle corresponding to the linear displacement of the pin driver?

3.83. Figure P3.84 shows a grasping device for a mechanical manipulator designed by Komatsu (utility model 1974-68167). It is powered by cylinder motion.
- **(a)** Calculate the degrees of freedom of this mechanism (disregard the spring).
- **(b)** Calculate the mechanical advantage (force exerted on the work/rod force) in the position shown.

3.84. Figure P3.85 shows mechanical grasping fingers for thin plates designed by Tadashi Aizawa (Japanese Patent 1974-36304). It is powered by a cylinder.
- **(a)** Calculate the degrees of freedom of this mechanism using Gruebler's equation.
- **(b)** Calculate the mechanical advantage (force exerted/piston force) in the position shown.
- **(c)** Calculate the mechanical advantage (force exerted/piston force) when grasping the plate.

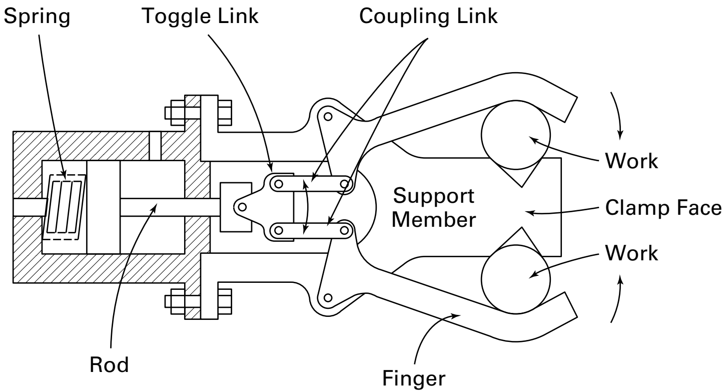


Figure P3.84

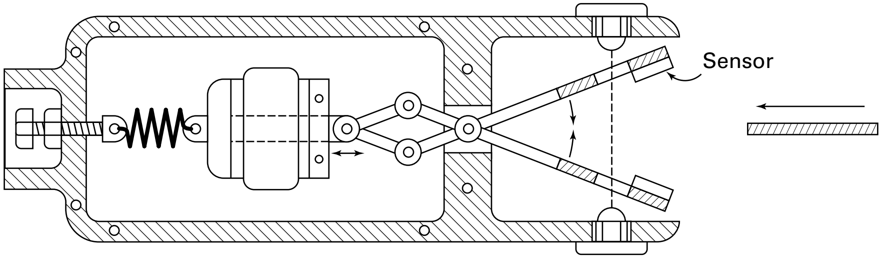


Figure P3.85